

# Comment on "Light-front Schwinger model at finite temperature"

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## Abstract

In a recent paper by A. Das and X. Zhou [Phys. Rev. D **68**, 065017 (2003)] it is claimed that explicit evaluation of the thermal photon self-energy in the Schwinger model gives off-shell thermal Green functions that are different in light-front and conventional quantizations. We show that the claimed difference originates from an erroneous simplification of the fermion propagator used in the light-front calculation.

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In a recent paper by A. Das and X. Zhou [1] it is claimed that explicit evaluation of the finite temperature photon self-energy in the Schwinger model shows that the photon off-shell thermal Green function is different in light front (LF) and conventional quantizations. It is also claimed that this could be a counterexample to the general proof of equivalence given by us in Ref. [2]. Here we show that the calculated difference obtained in Ref. [1] is due to an erroneous simplification of the expression for the zero mass fermion propagator, and in no way contradicts the proof of equivalence given in Ref. [2].

As is stated in Ref. [2], we have given a proof of equivalence between LF and conventional thermal field theories for those models which are equivalent at zero temperature. It is clear, therefore, that this does not imply that all LF Green functions are identical to their conventional counterparts, as this statement is not even correct at zero temperature (see for example Refs. [3, 4, 5, 6]). Yet any difference between thermal Green functions can always be traced to a difference at zero temperature; in other words, what we have shown is that temperature is never the origin of any non-equivalence.

The difference in the finite temperature off-mass-shell photon self-energy calculated in Ref. [1], however, has a totally different origin. In Ref. [1] the photon self-energy is calculated in the real time formalism. The LF real time fermion propagator derived in Ref. [1] is given by their Eq. (33), and in the zero mass limit is given by<sup>1</sup>

$$iS_{++}(\bar{p}) = \bar{\gamma}^\mu \bar{p}_\mu (\bar{\gamma}^0 - \bar{\gamma}^1) \left\{ \frac{i}{-(2\bar{p}_0 + \bar{p}_1)\bar{p}_1 + i\epsilon} - 2\pi n_F(|\bar{p}_0|)\delta[(2\bar{p}_0 + \bar{p}_1)\bar{p}_1] \right\}. \quad (1)$$

The error in Ref. [1], responsible for the difference between LF and conventional amplitudes, occurs when this propagator is simplified as

$$iS_{++}(\bar{p}) \rightarrow \bar{\gamma}^\mu \bar{p}_\mu (\bar{\gamma}^0 - \bar{\gamma}^1) \left\{ \frac{i}{-(2\bar{p}_0 + \bar{p}_1)\bar{p}_1 + i\epsilon} - P^+ 2\pi n_F(|\bar{p}_0|)\delta[(2\bar{p}_0 + \bar{p}_1)\bar{p}_1] \right\} \quad (2)$$

(Eq. (49) of Ref. [1]) where  $P^+ = -\frac{1}{2}\bar{\gamma}^1\bar{\gamma}^0$  is the projection operator on the LF fermion dynamical degrees of freedom,  $\psi_+ = P^+\psi$ , (see Eq. (25) in Ref. [1]). The simplification indicated by Eq. (2) is obtained by dropping the term where  $-\pi n_F(|\bar{p}_0|)$  is multiplied by

$$\begin{aligned} & 2\bar{\gamma}^\mu \bar{p}_\mu (\bar{\gamma}^0 - \bar{\gamma}^1) (1 - P^+)\delta[(2\bar{p}_0 + \bar{p}_1)\bar{p}_1] \\ & = -\bar{\gamma}^0\bar{\gamma}^1(2\bar{p}_0 + \bar{p}_1)\delta[(2\bar{p}_0 + \bar{p}_1)\bar{p}_1] \neq 0. \end{aligned} \quad (3)$$

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<sup>1</sup> Ref. [1] uses variables defined as  $\bar{p}^\mu = (\sqrt{2}p^+, p^1)$  and  $\bar{p}_\mu = (p^0, -\sqrt{2}p^+)$  where  $p^+ = \frac{1}{\sqrt{2}}(p^0 + p^1)$ .

That the term of Eq. (3) has been dropped suggests that the mass shell  $\delta$ -function has been treated as  $\delta[(2\bar{p}_0 + \bar{p}_1)\bar{p}_1] = \frac{1}{|\bar{p}_1|}\delta(2\bar{p}_0 + \bar{p}_1)$  which is right only if  $\bar{p}_1 \neq 0$ . In this way the part of the light-cone,  $\bar{p}_1 = 0$ , is missing. In the whole two-dimensional plane of variables  $\bar{p}_0$  and  $\bar{p}_1$ , the correct formula is

$$\delta[(2\bar{p}_0 + \bar{p}_1)\bar{p}_1] = \frac{1}{|\bar{p}_1|}\delta(2\bar{p}_0 + \bar{p}_1) + \frac{1}{2|\bar{p}_0|}\delta(\bar{p}_1). \quad (4)$$

In this respect, it is worth noting that although in the massive particle case ( $m \neq 0$ )

$$\delta[(2\bar{p}_0 + \bar{p}_1)\bar{p}_1 - m^2] = \frac{1}{|\bar{p}_1|}\delta(2\bar{p}_0 + \bar{p}_1 - \frac{m^2}{\bar{p}_1}), \quad (5)$$

it would be wrong to conclude from this that  $\delta[(2\bar{p}_0 + \bar{p}_1)\bar{p}_1 - m^2] \rightarrow \frac{1}{|\bar{p}_1|}\delta(2\bar{p}_0 + \bar{p}_1)$  in the zero mass limit.

Use of the propagator of Eq. (2) in the calculation of the thermal photon self-energy leads to an amplitude (Eq. (53) of Ref. [1]) whose nonanalyticity is due to just one  $\delta$  function,  $\delta(2\bar{p}_0 + \bar{p}_1)$ , which is only a part of the nonanalyticity obtained in the conventional approach where the term  $\delta(\bar{p}_1)$  is also present (but expressed in terms of variable  $p^+ = -\bar{p}_1/\sqrt{2}$ ) [7] (here  $\bar{p}_0$  and  $\bar{p}_1$  are the total momentum components of the loop). It is a simple exercise to restore the  $\delta(\bar{p}_1)$  term in the LF calculation of the self-energy by keeping the last term in Eq. (4) for the mass shell  $\delta$ -function. As in Ref. [1], let us consider the photon self-energy in the real time formalism, namely, its  $++$  component in the space of doubled degrees of freedom

$$\bar{\Pi}^{\mu\nu}(\bar{p}) = e^2 \int \frac{d^2\bar{k}}{(2\pi)^2} \text{Tr}(\bar{\gamma}^0 - \bar{\gamma}^1)\bar{\gamma}^\mu iS_{++}(\bar{k})(\bar{\gamma}^0 - \bar{\gamma}^1)\bar{\gamma}^\nu iS_{++}(\bar{k} + \bar{p}), \quad (6)$$

but with  $S_{++}$  given by Eq. (1) together with Eq. (4) (rather than by Eq. (2) as in Ref. [1]). We note that the  $\bar{\gamma}^\mu$  here are  $2 \times 2$  matrices [7]. Evaluating the temperature dependent part of Eq. (6), denoted by  $\bar{\Pi}^{\mu\nu(\beta)}(\bar{p})$ , one finds that

$$\bar{\Pi}^{00(\beta)}(\bar{p}) = 2\bar{\Pi}^{01(\beta)}(\bar{p}) = 2\bar{\Pi}^{10(\beta)}(\bar{p}) \quad (7)$$

where

$$\begin{aligned} \bar{\Pi}^{01(\beta)}(\bar{p}) = & -e^2 \delta(2\bar{p}_0 + \bar{p}_1) \int d\bar{k}_1 \text{sgn}(\bar{k}_1) \text{sgn}(\bar{k}_1 + \bar{p}_1) \\ & \times [n_F(|\bar{k}_1|/2) + n_F(|\bar{k}_1 + \bar{p}_1|/2) - 2n_F(|\bar{k}_1|/2)n_F(|\bar{k}_1 + \bar{p}_1|/2)], \end{aligned} \quad (8)$$

which is in agreement with Ref. [1], but only because these particular tensor components have momentum factors that nullify the contribution of the  $\delta(\bar{p}_1)$  term, even if it is retained in the calculation. However, such a momentum suppression of the  $\delta(\bar{p}_1)$  term does not occur for the tensor component  $\bar{\Pi}^{11(\beta)}(\bar{p})$ . To see this, we write

$$2\bar{\Pi}^{11(\beta)}(\bar{p}) = \bar{\Pi}^{01(\beta)}(\bar{p}) + \bar{\Pi}^{(\beta)}(\bar{p}) \quad (9)$$

so that

$$\begin{aligned} \bar{\Pi}^{(\beta)}(\bar{p}) = e^2 \int \frac{d^2\bar{k}}{2\pi} \text{Tr} (2\bar{\gamma}^1 - \bar{\gamma}^0) \bar{k} \bar{\gamma}^1 (\bar{k} + \not{p}) \left\{ \frac{-in_F(|\bar{k}_0 + \bar{p}_0|)\delta[(\bar{k} + \bar{p})^2]}{\bar{k}^2 + i\epsilon} \right. \\ \left. + \frac{-in_F(|\bar{k}_0|)\delta(\bar{k}^2)}{(\bar{k} + \bar{p})^2 + i\epsilon} + 2\pi n_F(|\bar{k}_0 + \bar{p}_0|)\delta[(\bar{k} + \bar{p})^2]n_F(|\bar{k}_0|)\delta(\bar{k}^2) \right\}, \quad (10) \end{aligned}$$

where  $\bar{k}^2 = -(2\bar{k}_0 + \bar{k}_1)\bar{k}_1$  and  $(\bar{k} + \bar{p})^2 = -[2(\bar{k}_0 + \bar{p}_0) + \bar{k}_1 + \bar{p}_1](\bar{k}_1 + \bar{p}_1)$ . Expressing the  $\delta$  functions as a sum of two terms, as in Eq. (4), we see that the trace,  $\text{Tr} (2\bar{\gamma}^1 - \bar{\gamma}^0) \bar{k} \bar{\gamma}^1 (\bar{k} + \not{p}) = 2(2\bar{k}_0 + \bar{k}_1) [2(\bar{k}_0 + \bar{p}_0) + \bar{k}_1 + \bar{p}_1]$ , nullifies some of these terms so that

$$\begin{aligned} \bar{\Pi}^{(\beta)}(\bar{p}) = e^2 \int \frac{d^2\bar{k}}{2\pi} (2\bar{k}_0 + \bar{k}_1) [2(\bar{k}_0 + \bar{p}_0) + \bar{k}_1 + \bar{p}_1] \left\{ \frac{-in_F(|\bar{k}_0 + \bar{p}_0|)\delta(\bar{k}_1 + \bar{p}_1)}{|\bar{k}_0 + \bar{p}_0|[-(2\bar{k}_0 + \bar{k}_1)\bar{k}_1 + i\epsilon]} \right. \\ \left. + \frac{-in_F(|\bar{k}_0|)\delta(\bar{k}_1)}{|\bar{k}_0|[-[2(\bar{k}_0 + \bar{p}_0) + \bar{p}_1]\bar{p}_1 + i\epsilon]} + \frac{2\pi n_F(|\bar{k}_0 + \bar{p}_0|)\delta(\bar{k}_1 + \bar{p}_1)n_F(|\bar{k}_0|)\delta(\bar{k}_1)}{2|\bar{k}_0 + \bar{p}_0||\bar{k}_0|} \right\}. \quad (11) \end{aligned}$$

After further simplification we obtain

$$\begin{aligned} \bar{\Pi}^{(\beta)}(\bar{p}) = -2e^2 \delta(\bar{p}_1) \int d\bar{k}_0 \text{sgn}(\bar{k}_0) \text{sgn}(\bar{k}_0 + \bar{p}_0) \\ \times [n_F(|\bar{k}_0|) + n_F(|\bar{k}_0 + \bar{p}_0|) - 2n_F(|\bar{k}_0|)n_F(|\bar{k}_0 + \bar{p}_0|)]. \quad (12) \end{aligned}$$

It is now easy to check that these results for  $\bar{\Pi}^{00(\beta)}(\bar{p})$ ,  $\bar{\Pi}^{10(\beta)}(\bar{p})$ ,  $\bar{\Pi}^{01(\beta)}(\bar{p})$ , and  $\bar{\Pi}^{11(\beta)}(\bar{p})$ , calculated using the LF formalism of Ref. [1], are identical to those calculated in the conventional case [7].

Although the above analysis is straightforward, it needs to be recognized that the taking of the zero mass limit in 1 + 1 dimensional LF field theory is well-known for its subtlety. Thus, although the right hand side of Eq. (4) is the correct smooth massless limit of Eq. (5), we would like to confirm, from the underlying fundamental quantum field theory of massless fermions, that it is just this smooth massless limit that is required to obtain the correct massless fermion propagator in the Schwinger model. For this purpose, we reexamine the

above analysis from the point of view of the first consistent formulation of this problem worked out by McCartor some 15 years ago [8].

McCartor found that the correct way to quantize massless fields on the LF in 1+1 dimensions is quite different from the way used to quantize massive fields (see below). From this fact one might be tempted to conclude that the massless propagator cannot be the smooth massless limit of the massive propagator; yet, as we shall see, it is. Following McCartor's quantization scheme, one obtains (see below) the following LF real time propagator for a massless fermion:

$$D^L(p) = \frac{\pi}{2} \text{sgn}(p^-) \gamma^+ \delta(p^+) + \frac{i\gamma^- p^+}{p^2 + i\epsilon} - \pi \text{sgn}(p^0) n_F(|p_0|) [\gamma^+ \delta(p^+) + \gamma^- \delta(p^-)] \quad (13)$$

where  $p^\pm = \frac{1}{\sqrt{2}}(p^0 \pm p^1)$  and  $\gamma^\pm = \frac{1}{\sqrt{2}}(\gamma^0 \pm \gamma^1)$ . For easy comparison with the expressions of Ref. [1], we note that  $p^+ = -\frac{1}{\sqrt{2}}\bar{p}_1$ ,  $p^- = \frac{1}{\sqrt{2}}(2\bar{p}_0 + \bar{p}_1)$ ,  $p^0 = \bar{p}_0$ , and  $p^2 = -\bar{p}_1(2\bar{p}_0 + \bar{p}_1)$ ; also, the propagators of Ref. [1], and those denoted by  $S_{++}$  above, are defined with an extra factor of  $\gamma^0 = \bar{\gamma}^0 - \bar{\gamma}^1$  compared to those denoted by  $D$ 's above, below, and in Ref. [2]. Some interesting observations can now be made. Firstly, it is evident that the temperature dependent part of the propagator, the last term of Eq. (13), is exactly the same as in the conventional propagator, given by

$$D(p) = \not{p} \left[ \frac{i}{p^2 + i\epsilon} - 2\pi n_F(|p_0|) \delta(p^2) \right]. \quad (14)$$

This is consistent with our observation in Ref. [2], and mentioned above, that temperature is never the source of non-equivalence between LF and conventional theories. Secondly, the LF propagator of Eq. (13) coincides with the smooth massless limit of the LF spinor propagator of Refs. [2] and [9],

$$D^L(p, m) = (\not{p}_{\text{on}} + m) \left[ \frac{i}{p^2 - m^2 + i\epsilon} - 2\pi n_F(|p_0|) \delta(p^2 - m^2) \right] \quad (15)$$

where  $p_{\text{on}}$  is the on mass shell momentum:  $p_{\text{on}}^- = m^2/2p^+$ ,  $p_{\text{on}}^+ = p^+$ . Note, that in order to obtain this coincidence, the massless limit has to be taken in the mathematically correct way, which means that the  $\delta$  function in Eq. (15),  $\delta(p^2 - m^2) = \delta[\bar{p}_1(2\bar{p}_0 + \bar{p}_1) + m^2]$ , must reduce in the massless limit to a sum of two  $\delta$  functions as given by Eq. (4).

Comparing now the propagator of Eq. (1) with the ones denoted with  $D$ 's above, we see that it is identical with the conventional propagator given in Eq. (14). Our finding that the propagator of Eq. (1) gives photon self-energies that are identical with those calculated in

conventional theory, is therefore not surprising. What is at issue, however, is the proper way to take the zero mass limit, and this is answered by the straightforward reduction of Eq. (15) into Eq. (13), as just discussed.

To complete our analysis, we outline the derivation of Eq. (13). The LF quantization of massless fields in  $1 + 1$  dimensions, formulated in Ref. [8], prescribes the following anticommutators of fermion fields on the hyperplanes  $x^+ = 0$  and  $x^- = 0$ :

$$\{\psi_+(x), \psi_+^\dagger(0)\}_{x^+=0} = \frac{1}{\sqrt{2}} P^+ \delta(x^-) \quad (16a)$$

$$\{\psi_-(x), \psi_-^\dagger(0)\}_{x^-=0} = \frac{1}{\sqrt{2}} P^- \delta(x^+) \quad (16b)$$

where  $P^\pm = \frac{1}{2}(1 \pm \gamma^0 \gamma^1)$  are the projection operators (the same  $P^+$  operator was used previously in Eq. (2)), and  $\psi_\pm = P^\pm \psi$  are the dynamical (+) and non-dynamical (−) field components. In combination with the free field equations  $\partial\psi_+(x)/\partial x^+ = 0$  and  $\partial\psi_-(x)/\partial x^- = 0$ , one obtains the anticommutator in the entire space

$$\{\psi(x), \bar{\psi}(0)\} = \frac{1}{2} [\gamma^+ \delta(x^+) + \gamma^- \delta(x^-)] . \quad (17)$$

This commutator is just the free LF spectral function whose Fourier transform is

$$\rho_F^L(p) = \pi [\gamma^+ \delta(p^+) + \gamma^- \delta(p^-)] . \quad (18)$$

Now the spinor particle propagator, defined as the ensemble average of the  $x^+$ -time ordered product, can be written in terms of the LF Lehmann representation as [2]

$$D^L(p) = i \int \frac{dp'_0}{2\pi} \frac{\rho^L(p'_0, p^+)}{p_0 - p'_0 + i\epsilon} - n_F(p_0) \rho^L(p). \quad (19)$$

In the free case, substituting  $\rho^L(p) = \rho_F^L(p)$  as given by Eq. (18), one gets Eq. (13). This completes our analysis and allows us to make a number of further observations.

Firstly, the above analysis shows explicitly that in LF thermal field theory of massless spinors in  $1 + 1$  dimensions, both the dynamical and non-dynamical components get thermalized. This is in contradiction to the assertions made in Refs. [1, 10, 11]. Although the analysis presented here was for real time propagators, the same conclusion is reached when a similar analysis is made of imaginary time propagators [12].

Secondly, taking the smooth massless limit applies not just to propagators but to spectral functions as well. For example, the free spectral function of Eq. (18) is just the smooth

massless limit of the free LF spinor spectral function of Ref. [2],

$$\rho_F^L(p, m) = 2\pi \operatorname{sgn}(p_0) (\not{p} + m) \delta(p^2 - m^2). \quad (20)$$

In this regard, although the part of Eq. (18) defined as  $\rho_F^{L-}(p) \equiv \pi \gamma^- \delta(p^-)$  cannot be considered as the massless limit of Eq. (20), one could nevertheless consider it as a "prescription of quantization" and use it in Eq. (19). One then obtains

$$D^L(p) = \frac{i\gamma^- p^+}{p^2 + i\eta} - n_F(|p_0|) \pi \epsilon(p^0) \gamma^- \delta(p^-) \quad (21)$$

whose temperature dependent part is different from the temperature dependent part of the conventional propagator. This, of course, is just what has been argued in Refs. [1, 10, 11]. To understand the origin of this nonequivalence we examine how such a propagator can arise from a field theoretic point of view. In this regard we note that the spectral function  $\rho_F^L(p) = \pi \gamma^- \delta(p^-)$  corresponds to quantization specified by anticommutators:

$$\{\psi_+(x), \psi_+^\dagger(0)\}_{x^+=0} = \frac{1}{\sqrt{2}} P^+ \delta(x^-), \quad \{\psi_-(x), \psi_-^\dagger(0)\}_{x^-=0} = 0. \quad (22)$$

Just these commutators were in fact used for quantization prior to the publication of McCartor's work, and describe physics which is not equivalent to the Schwinger model, as explained in Ref. [8]. In this light, one can see that the calculation of the temperature dependent part of the photon self-energy diagram, as given in Ref. [1] and discussed in Ref. [11], corresponds to a treatment which is not only not equivalent to the conventional approach at finite temperature, as claimed in Refs. [1, 10, 11], but it is not equivalent even at zero temperature.

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